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Using stochastic-mechanical methods borrowed from quantum physics, we study the average shape of an acoustic pulse from a source on the SOFAR axis as a function of range. As shown by Williamson, this shape is given by the graph of the probability density of travel time along a random ray. We sketch a possible approach to finding this probability density, and give a formula for its average value which predicts the known transition from near-range to far-range pulse shape.

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A STOCHASTIC-MECHANICAL APPROACH TO ACOUSTIC PULSE PROPAGATION IN THE DEEP OCEAN SOUND CHANNEL*

by

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ABSTRACT

We study the shape of an acoustic pulse from a monochromatic source on the SOFAR axis as a function of range in the parabolic approximation. Borrowing a method of quantum physics, we work with the stochastic-mechanical interpretation of the parabolic wave equation: the motion of acoustic "particles" along random rays is governed by a stochastic differential equation of diffusion type. As recognized by Williamson in his study of electromagnetic pulses, the average pulse shape is just the (suitably normalized) graph of the probability density function of a particle's travel time along a random ray. We discuss an approach to finding this probability density at long range, and we derive a formula for the average travel time as a function of range; this formula predicts the well-known transition from the characteristic near-range pulse shape to the characteristic far-range shape. Knowledge of a key parameter in this formula is equivalent to knowing the location of convergence zones. A possible test of this theory against experimental data is outlined. In the course of the paper, we derive new stochastic-mechanical versions of Ehrenfest's theorem and the virial theorem.

A STOCHASTIC-MECHANICAL APPROACH TO ACOUSTIC PULSE PROPAGATION IN THE DEEP OCEAN SOUND CHANNEL

Introduction

The effects of inhomogeneities of the medium on the propagation of pulses of acoustic or electromagnetic energy are of fundamental importance for a wide range of scientific and engineering work, including the design and operation of radar and sonar detection systems, seismic and ionospheric probes, radio telescopes, and communication systems. Describing the spreading of a pulse as it travels, and the shape of a pulse as a function of time at various ranges, are basic problems. For example, pulse spreading limits the ability of zero-forcing equalizers to reduce intersymbol interference in communication systems ([1], chapter 9). Astonomers have extensively investigated the spreading of radio signals from pulsars ([2], [3], and references therein).

In applied underwater acoustics, the use of sound pulses extends from the earliest efforts to locate airmen lost at sea to the latest attempts to map the fine features of the ocean floor with acoustic probes. As is well known, the deep sound channel, or SOFAR channel, with axis at the depth of minimum sound speed in the ocean, acts as an acoustic waveguide, so that sound from explosions near the channel, and other strong signals entering the channel, can propagate to distances exceeding 3000 kilometers. Pulse propagation in the SOFAR channel therefore

has, among others, applications on a global scale: the monitoring of underwater volcanic activity and earthquakes to anticipate the potentially catastrophic accompanying tidal waves. The importance of pulse propagation in these and other applications of underwater sound, including sonar, has recently stimulated vigorous theoretical and experimental investigation of the phenomenon. For accounts of several recent projects, see [4]; a more recent study is [5].

In the present paper, we are concerned with acoustic pulse propagation in the SOFAR channel; in particular, we are interested in pulse spreading and pulse shape as a function of range for a monochromatic source on the SOFAR axis. One of the first features of this phenomenon to be recognized is the radical alteration of pulse shape as the pulse moves from the region near the source to ranges far from the source. This transformation of the shape of the graph of the time-varying acoustic intensity received down range on the sound axis is well-illustrated by the two records of acoustic signals shown below, which were first published in 1949 [6] and have been reproduced in Brekhovskikh's book([7], p.504):

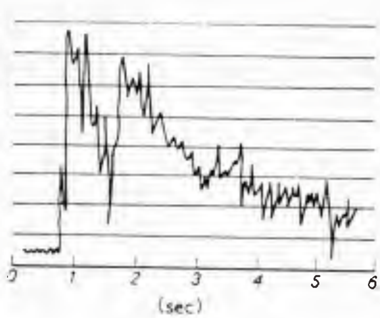


FIG. A. Near Range

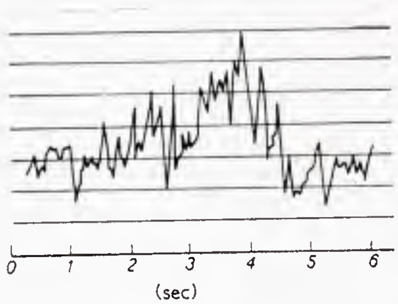


FIG. B. Far Range

These pictures are to be contrasted with FIG. C below, a schematic rendering of a theoretical pulse shape for electromagnetic radiation due to Williamson [2]:

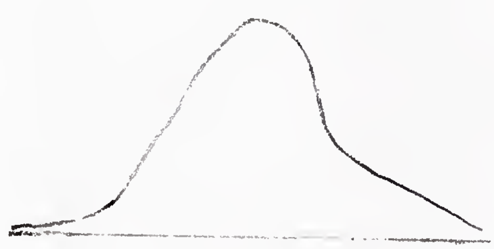


FIG. C.

In comparing these figures, note that any record after five seconds in FIGS. A and B is of background noise.

The differences in figures A, B, C are well-understood qualitatively. At short range in FIG. A, the direct ray along the sound channel axis, which carries most of the acoustic energy, arrives before the refracted rays, which account for the (more or less) exponentially decreasing tail. However, at long range, in FIG. B, the refracted rays, even though they have traveled over longer distances than the direct ray, have done so

at faster sound speeds for a long enough time to overtake the direct ray. Hence, we find a (more or less) exponential increase to maximum intensity, followed by an abrupt decrease to zero intensity. FIG. C is different from both FIGS. A and B. The assumptions made in deriving FIG. C are that all rays are refracted to some extent, but that the speed of light is constant throughout the medium (as seems reasonable for the scattering of light rays by interstellar dust, for example). Thus, there is no direct, or unrefracted, ray; if there were one it would arrive at the extreme left of FIG. C, but all the photons travel along refracted rays and are delayed somewhat. There is a gradual buildup of intensity to a maximum, followed by a gradual, near-exponential decrease to zero intensity. We see, then, that the distinctive pulse shapes of FIG. A and FIG. B, and the transformation of FIG. A into FIG. B with increasing travel time of the pulse are accounted for by the variation of the speed of sound with depth in the ocean, and the focusing of acoustic rays by the ocean sound channel at the depth of minimum sound speed on the channel axis.

In this paper we propose to investigate FIGS. A and B quantitatively. Since we are studying propagation in the sound channel, we work in the parabolic approximation (e.g., [8]). The domain of validity of this approximation depends on the wavelength, but it is essentially a long-range approximation; hence, our methods are more justifiably applied to FIG. B. Mathematically, the parabolic approximation is embodied in the

parabolic wave equation, which is identical to the Schrodinger equation of quantum mechanics. We therefore expect that ways of analyzing and interpreting the Schrodinger equation will be relevant to our problem. For example, Dashen [4, 9] has recently shown that Feynman path integrals [10, 11] are a useful tool in analyzing wave propagation in random media in the parabolic approximation.

The present author has recently initiated a line of acoustic research [12] based on another interpretation of the Schrodinger equation, Nelson's stochastic mechanics. (For early work in stochastic mechanics, see [13-15]. Two recent review articles, with references to subsequent work, are [16] and [17]; see also [18].) Just as Williamson in his work cited above on electromagnetic radiation developed a picture of a flux of photons along stochastic ray paths, we work with the stochastic mechanical picture of acoustic "particles" (or "bundles of energy") traveling along random rays from source to receiver. This picture is especially apt for analyzing pulse shapes because, as Williamson recognized and exploited, with proper normalization the average pulse shape is just the graph of the probability density function of the travel time along the random rays. According to [3], Williamson achieved the first theoretical prediction of average shapes of scattered electromagnetic pulses by determining this probability density function.

A main goal of our research has accordingly been to derive the analogous acoustic result, i.e., to find the probability distribution of the travel time along random rays in the stochastic mechanical picture; or, more modestly, to find the asymptotic distribution as the range approaches infinity. The graph of this probability density function (suitably scaled, and reflected about the vertical axis to take into account the overtaking of the direct ray by the refracted ones) would then be the average of records like FIG. B in the far-field region (whose demarcation would, as usual, depend on the wavelength). We have so far not achieved this goal. One possible approach to this problem is discussed at the end of this paper.

We do present in this paper a formula (eq. (47) below) for the average travel time along random rays that predicts the transition from pulses like FIG. A to those like FIG. B as the range increases. More precisely, if $t_o(z)$ is the time for the direct ray to travel to range z, and $E(t(z))$ is the travel time to range z averaged over all random rays, then we derive a formula for $E(t(z))$ that indicates that $E(t(z)) > t_o(z)$ for small z and $E(t(z)) < t_o(z)$ for large z. Our formula is a simple one containing three parameters. Evaluation of these parameters would in principle allow our formula to be used for the location of convergence zones (where $E(t(z)) = t_o(z)$), a topic of some current research interest [19]; conversely, information about the location of convergence zones could be used to test our theory.

We wish to point out that the very definition of the notion of travel time along a stochastic mechanical ray path presents a fundamental problem which itself is currently under investigation: how, if at all, is one to define arclength for paths which are nowhere differentiable ([20], p.18) and even fail to be of bounded variation in any interval ([21], p.395)? A possible solution to this problem has recently been suggested by Zambrini and Yasue [22], and we make use of their suggestion below. It will be clear below that the choice of definition of arclength is crucial for the work described in this paper; moreover, we believe that it may not be an exaggeration to say that the ultimate resolution of the problem will play a large role in determining the future applications of stochastic mechanics. We mention in passing that the equivalent question of defining the angle between a ray and the horizontal direction has recently been treated by Palmer [23] in the framework of Dashen's path integral theory.

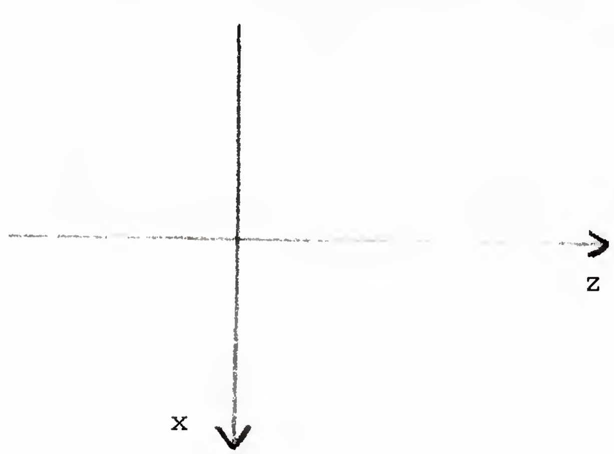
In our present work we choose a deterministic sound speed profile which is a good approximation to the "canonical" profile of Munk ([24]; [4], p.32) near the axis of the sound channel. the random character of the rays then results solely from their mutual diffractive interference; indeed, it is suggestive to think of the stochastic mechanical picture as introducing into geometric acoustics the diffractive effects retained in the parabolic approximation [12], such an interpretation being analogous to the semi-classical interpretation of quantum

mechanics. It will be seen, however, that our methods can be applied also in the case of a randomly fluctuating sound speed profile; our methods are "honest" in the sense of Keller [25].

A sketch of the plan of the paper follows. In part I we establish notation and review some salient features of the parabolic approximation. In part II we present the equations from stochastic mechanics that we need. Two of these, Ehrenfest's Theorem and the virial theorem, are new stochastic mechanical versions of well-known classical and quantum mechanical principles; we expect they will consequently be of independent interest. In part III, we define the sound speed profile, arclength of a random ray path, and travel time along such a path. In part IV we derive our formula for average travel time. In part V we discuss the prediction of average pulse shape at long ranges and other possible extensions, refinements, and applications of the present work.

I. Parabolic Approximation

Ignoring azimuthal variations, we work in the two-dimensional coordinate system shown below, in which z is the range coordinate, increasing to the right, and x is the depth coordinate, increasing in the downward direction.



We choose x=0 to correspond to the axis of the deep ocean sound channel. In the parabolic approximation, the evolution of the acoustic pressure field is described in terms of the propagation of a complex-valued envelope $\Psi = \Psi(x,z)$ which satisfies the parabolic wave equation [8]:

$$\frac{i}{k}\frac{\partial \Psi}{\partial z} = -\frac{1}{2k^2}\frac{\partial^2 \Psi}{\partial x^2} + \mu(x)\Psi , \tag{1}$$

where k is the wavenumber, assumed constant, and

$$\mu(x) = 1 - n(x), \tag{2}$$

n(x) being the depth-dependent index of refraction:

$$n(x) = C_o/C(x), \tag{3}$$

with C(x) equal to the velocity of sound at depth x, and $C_o \equiv C(o)$ equal to the minimum velocity of sound on the sound channel axis. Notice that with these conventions, μ is non-negative. Equation (1) is identical to the Schrodinger equation for a particle of unit mass if we identify 1/k with Planck's constant divided by 2π, $\mu(x)$ with the potential energy, and the range coordinate z with the time.

The replacement of the ordinary wave equation by (1) rests on several assumptions valid for long-range propagation in the sound channel, including the assumption of gentle refraction

$$n \cong 1, \tag{4}$$

which means that μ in (2) is a small quantity, and the small-angle condition

$$\theta << 1 \text{ radian}, \tag{5}$$

where θ is the angle between a ray's direction at any point on it and the positive z-axis. Since the ocean sound channel acts like

a vertically-thin, horizontally elongated waveguide, (5) is a reasonable assumption for rays which propagate in the channel to large distances. We have mentioned above the difficulty of defining θ in our theory, but we will still find that (5) has heuristic value.

The parabolic approximation retains diffractive effects, and hence is valid to significantly lower frequencies than geometric acoustics. Unlike normal mode expansions, the parabolic approximation can accomodate range-dependent inhomogeneities: we could assume $n = n(x,z)$ and $\mu = \mu(x,z)$ above, corresponding to the consideration of time-dependent potentials in quantum mechanics. The stochastic mechanical equations given below would have to be changed in a seemingly straightforward way. We do not pursue this possibility further in this paper.

II. Stochastic Mechanical Formulas:

In this section we present a brief account of some basic assumptions and results of stochastic mechanics sufficient for our needs in parts III and IV; we do not aim for completeness, but refer the interest reader to fuller accounts in the literature [13-15, 18]. We do give derivations from first principles of two results familiar in quantum mechanics but, as far as we know, new in stochastic mechanics: Ehrenfest's theorem (cf. [26], p.216 ff.) and the virial theorem (cf. [27], p.168). We also give a new proof of Nelson's energy conservation Law [18], a comparatively recent stochastic mechanical result that we

often use below in parts III and IV. Our argument is actually a converse to Nelson's argument in [18], where he derives the basic dynamical equation (16) below from the law of energy conservation; we derive the energy conservation law from (16) and other basic stochastic mechanical equations.

It must be pointed out that, because the role of the time variable in the Schrodinger equation is now being played by the spatial variable z in (1), the variables below with their customary stochastic mechanical names of "velocity" and "energy" are actually dimensionless here, and so must not be uncritically identified with the actual velocities and energy of physical particles; the introduction of a natural time scale would eliminate this distinction, but we have not needed to take this step.

Consider a particle moving along a random ray path $z \to x(z)$; here $x(z)$ denotes the depth of the particle at range z. This depth $x(z)$ is a real-valued random variable; more specifically, the random process $z \to x(z)$ is assumed to be a diffusion, so that $x(z)$ is governed by an Ito stochastic differential equation [28-30]:

$$dx(z) = b(x(z),z)dz + \sqrt{2\nu}\, dw(z), \tag{6}$$

where b is called the drift, ν is a constant called the diffusion coefficient, and $z \to w(z)$ is standard Brownian motion (also called the Wiener process). Suppose $f(x,z)$ is a smooth function which does not increase too rapidly as $x \to \pm\infty$. (It is

difficult to specify exactly how rapidly--see ([14], p.103)--because the relevant a priori estimates for solutions of the parabolic wave equation remain, as far as we know, unknown; however, roughly speaking, it suffices that any low order partial derivative of f multiplied by any low order partial derivative of the density ρ defined below approach zero so x approaches $\pm\infty$. As a practical matter, we can safely assume such behavior for any acoustic quantity for sound channel propagation.) Following Nelson, we define the mean forward derivative $Df(x(z),z)$ of $f(x(z),z)$ by

$$Df(x(z),z) = \lim_{\Delta z \downarrow o} E\left\{\frac{f(x(z+\Delta z),z+\Delta z) - f(x(z),z)}{\Delta z} \middle| P_z\right\}, \tag{7}$$

where $E\{Y|P_z\}$ denotes conditional expectation of the random variable Y with respect to the past P_z of the x-process at range z; i.e., the best estimate of Y we can make from information about x(s) for $0 \leq x \leq z$.

The time reversal of a diffusion process is again a process of the same type, so we have

$$dx(z) = b_*(x(z),z)dz + \sqrt{2\nu}\ dw(z)\ , \tag{8}$$

for a suitable backwards drift b_*; and we define the mean backward derivative $D_*f(x(z),z)$ of $f(x(z),z)$ by

$$D_*f(x(z),z) = \lim_{\Delta z \downarrow o} E\left\{\frac{f(x(z),z) - f(x(z-\Delta z),z-\Delta z)}{Dz} \middle| F_z\right\}, \tag{9}$$

where $E\{Y|F_z\}$ denotes conditional expectation of the random variable with respect to the future F_z of the x-process at range z.

Now, it follows from Ito's lemma ([29], p.33) that

$$Df(x(z),z) = \frac{\partial f}{\partial z}(x(z),z) + b(x(z),z)\frac{\partial f}{\partial x}(x(z),z) + \nu\frac{\partial^2 f}{\partial x^2}(x(z),z) \qquad (10)$$

and

$$D_* f(x(z),z) = \frac{\partial f}{\partial z}(x(z),z) + b_*(x(z),z)\frac{\partial f}{\partial x}(x(z),z) - \nu\frac{\partial^2 f}{\partial x^2}(x(z),z). \qquad (11)$$

In particular, for $f(x,z) = x$, we get

$$Dx(z) = b(x(z),z) \qquad (12)$$

and

$$D_* x(z) = b_*(x(z),z). \qquad (13)$$

Following Nelson [13, 14], we define the stochastic acceleration $a(x(t))$ by

$$a(x(t)) = \tfrac{1}{2}\{DD_* + D_* D\}(x(t)). \qquad (14)$$

Suppose also that we are given a potential energy function $\mu(x)$, and so a force $F(x) = -\frac{\partial \mu}{\partial x}$. Then a short computation using (10)-(14) shows that Newton's Second Law (recall we are taking m=1 whenever a mass m of a particle appears)

$$F(x(t)) = a(x(t)) \qquad (15)$$

is equivalent to

$$-(\nu\frac{\partial^2 u}{\partial x^2} + u\frac{\partial u}{\partial x}) + (\frac{\partial v}{\partial z} + v\frac{\partial v}{\partial x}) + \frac{\partial \mu}{\partial x} = 0, \qquad (16)$$

where

$$u = (b-b_*)/2 \qquad (17)$$

is called the osmotic velocity (for historical reasons -- see [14], p.21), and

$$v = (b+b_*)/2 \qquad (18)$$

is called the current velocity. In equations (16)-(18), we have neglected to indicate that all functions are evaluated at $x(z)$ and z (except $\mu = \mu(x(z))$, which we are assuming has no explicit dependence on z); for the sake of legibility, we shall often make the corresponding omissions below when no misunderstanding can result. Equation (16), our stochastic version of Newton's second law, is the basic dynamical equation of stochastic mechanics.

There is also a basic kinematical equation, which follows from the forward and backward Fokker-Planck equations and the fact that (7) and (8) are time reversals of one another:

$$\frac{\partial u}{\partial z} = -\nu \frac{\partial^2 v}{\partial x^2} - \frac{\partial}{\partial x}(uv). \tag{19}$$

It also is a kinematical fact that

$$u = \frac{\nu}{\rho} \frac{\partial \rho}{\partial x} , \tag{20}$$

where $\rho(x,z)$ is the probability density function of $x(z)$. Assuming ρ never equals zero, (20) can be written

$$u = 2\nu \frac{\partial R}{\partial x} \tag{21}$$

for $R = \frac{1}{2} \log \rho$. Also write v as a gradient: choose a function S so that

$$v = 2\nu \frac{2S}{\partial x}. \tag{22}$$

If we now define Ψ by

$$\Psi = e^{R+iS}, \tag{23}$$

and choose

$$\nu = \frac{1}{2k} , \tag{24}$$

then a simple calculation using (16) and (19) shows that Ψ satisfies the parabolic wave equation (1). Conversely, if Ψ

satisfies (1), and R, S, u, v are defined by (21)-(23), then u and v satisfy (16) and (19). In this sense, the stochastic mechanics embodied in (16) and (19) is equivalent to the parabolic wave equation.

We also have an important formula relating the ordinary z-derivative of average quantities to averages of their mean forward and mean backward derivatives ([14], p.98; [15], p.204): if f(x,z) and g(x,z) are smooth functions, not increasing too rapidly as $x \to \pm\infty$, and if "E" denotes averaging over all random rays $z \to x(z)$, then

$$\frac{d}{dz}E(f(x(z),z)g(x(z),z))$$
$$= E[(Df(x(z),z))(g(x(z),z)) + (f(x(z),z))(D_*(g(x(z),z)))]. \qquad (25)$$

In particular, with $g \equiv 1$ this equation yields

$$\frac{d}{dz}E(f(x(z),z)) = E(DF(x(z),z)) \ , \qquad (26)$$

and, interchanging f and g and putting $g \equiv 1$, we find

$$\frac{d}{dz}E(f(x(z),z)) = E(D_*f(x(z),z)), \qquad (27)$$

If we average (26) and (27), and then use (10), (11) and (18), we find

$$\frac{d}{dz}E(f(x(z),z)) = E(\frac{\partial f}{\partial z}(x(z),z) + v(x(z),z)\frac{\partial f}{\partial x}(x(z),z)). \qquad (28)$$

Similarly, if we subtract (27) from (26), and then use (10), (11) and (17), we obtain

$$E(u(x(z),z)\frac{\partial f}{\partial x}(x(z),z) + \nu \frac{\partial^2 f}{\partial x^2}(x(z),z)) = 0, \qquad (29)$$

and, replacing $\frac{\partial f}{\partial x}$ by f, we have

$$E(u(x(z),z)f(x(z),z) + \nu \frac{\partial f}{\partial x}(x(z),z)) = 0 . \qquad (30)$$

We now use (28)-(30), together with the basic equations (16) and (19), to derive Ehrenfest's theorem, the virial theorem, and the law of energy conservation. We shall often suppress the arguments x(z) and z below, as in equation (16) ff.

Taking the average of (16) and using (29) with f=u yields

$$E(\frac{\partial v}{\partial z} + v\frac{\partial v}{\partial x}) + E(\frac{\partial \mu}{\partial x}) = 0 ,$$

or, by (28) with f=v,

$$\frac{d}{dz}E(v) = E(- \frac{\partial \mu}{\partial x}).$$

But (28) with f=x yields

$$\frac{d}{dz}E(x) = E(v) , \qquad (31)$$

so we obtain <u>Ehrenfest's Theorem</u>:

$$\frac{d^2}{dz^2}E(x(z)) = E(- \frac{\partial \mu}{\partial x}(x(z))). \qquad (32)$$

This equation says that the average ray $z \to E(x(z))$ feels the average force, which is to be distinguished from $-\frac{\partial \mu}{\partial x}(E(x(z)))$, the force along the average ray. These forces are distinct, and the average path is consequently not a classical-mechanical path, unless the function $\frac{\partial \mu}{\partial x}$ is affine; i.e., unless the potential μ is at most a quadratic function of x.

To derive the virial theorem, begin by multiplying (16) by x and averaging:

$$-E(\nu x\frac{\partial^2 u}{\partial x^2} + ux\frac{\partial u}{\partial x}) + E(x\frac{\partial v}{\partial z}) + E(xv\frac{\partial v}{\partial x}) + E(x\frac{\partial \mu}{\partial x}) = 0. \qquad (*)$$

Now (29) with f=xu yields

$$E(ux\frac{\partial u}{\partial x} + u^2 + \nu x\frac{\partial^2 u}{\partial x^2} + 2\nu\frac{\partial u}{\partial x}) = 0,$$

and (30) with f=u gives

$$2\nu E(\frac{\partial u}{\partial x}) = -2E(u^2) \;;$$

substituting the second of these two equations into the first, we find

$$E(\nu x\frac{\partial^2 u}{\partial x^2} + ux\frac{\partial u}{\partial x}) = E(u^2) \;,$$

and using this in (*), we get

$$E(x\frac{\partial v}{\partial z} + vx\frac{\partial v}{\partial x}) = E(u^2) - E(x\frac{\partial \mu}{\partial x}) \;.$$

However, (28) with f=xv implies

$$E(x\frac{\partial v}{\partial z} + vx\frac{\partial v}{\partial x}) = \frac{d}{dz}E(xv) - E(v^2) \;,$$

so comparing the last two equations, we have the <u>virial theorem</u>:

$$\frac{d}{dz}E(xv) = E(u^2 + v^2) - E(x\frac{\partial \mu}{\partial x}) \;. \tag{33}$$

From (28) applied to $f(x) = \frac{1}{2}x^2$, we see that we can write the virial theorem in the alternate form

$$\frac{d^2}{dz^2}E(\tfrac{1}{2}x^2) = E(u^2 + v^2) - E(x\frac{\partial \mu}{\partial x}) \;. \tag{34}$$

Finally, we derive the <u>energy conservation law</u>:

$$E(\tfrac{1}{2}u^2 + \tfrac{1}{2}v^2 + \mu) = K \;, \tag{35}$$

when K is some constant (the total energy). We must show that

$$\frac{d}{dz}E(\tfrac{1}{2}u^2 + \tfrac{1}{2}v^2 + \mu) = 0 \;,$$

or, by (28), that

$$E(v\frac{\partial v}{\partial z} + v^2\frac{\partial v}{\partial x} + v\frac{\partial \mu}{\partial x}) = -E(u\frac{\partial u}{\partial z} + uv\frac{\partial u}{\partial x}) \quad . \tag{\&}$$

We will prove (&) by showing that each side of this equation equals $-\nu E(\frac{\partial u}{\partial x}\ \frac{\partial v}{\partial x})$. Examining the average of equation (16) multiplied through by v, we find that the left-hand side of (&) equals

$$E(\nu v\frac{\partial^2 u}{\partial x^2} + uv\frac{\partial u}{\partial x}) \ ;$$

applying (30) to the second term transforms this expression into

$$\nu E(v\frac{\partial^2 u}{\partial x^2} - \frac{\partial}{\partial x}(v\frac{\partial u}{\partial x}))$$

which is just $-\nu E(\frac{\partial u}{\partial x}\ \frac{\partial v}{\partial x})$, as claimed.

Turning to the right-hand side of (&), we see that the average of (19) multiplied through by u shows it to be equal to

$$E(\nu u\frac{\partial^2 v}{\partial x^2} + u^2\frac{\partial v}{\partial x}) \ ;$$

applying (30) to the second term yields

$$\nu E(u\frac{\partial^2 v}{\partial x^2} - \frac{\partial}{\partial x}(u\frac{\partial v}{\partial x})) = -\nu E(\frac{\partial u}{\partial x}\ \frac{\partial v}{\partial x}) \quad .$$

Therefore (&) is proved, and consequently so is (35).

III. Travel Time, Arclength, Index of Refraction

If one tries to use the usual expression

$$t(z) = \int_0^z \frac{d\sigma(z)}{C(x(z))} = \frac{1}{C_o}\int_0^z n(x(z))d\sigma(z) \tag{36}$$

to define the time $t(z)$ required for a particle traveling along the random ray $x(z)$ to go from range 0 to range z, one is confronted with the problem of giving meaning of the element of arclength $d\sigma$ for the nondifferentiable, nonrectifiable ray. Of course, we should have

$$(d\sigma)^2 = (dx)^2 + (dz)^2 , \tag{37}$$

but the usual formula $dx = (\frac{dx}{dz})dz$ is inapplicable because the derivative $\frac{dx}{dz}$ doesn't exist. We solve this problem in the manner of Zambrini and Yasue [22], who have recently considered this situation and proposed the definition

$$(dx)^2 = \frac{(u^2+v^2)}{L^2}(dz)^2 , \tag{38}$$

where L is a certain positive constant. They choose the value L=2 by requiring that $(dx)^2$ in (38) have its ordinary meaning in the limiting case of non-random differentiable trajectories; however, we prefer to leave L as a parameter in the theory. In our case the small angle condition (5) would suggest that $(dx)^2 << (dz)^2$, so that, because of (35), we should require

$$L^2 >> K . \tag{39}$$

Using formulas provided below, it would seem feasible to estimate the parameters K, L from empirical data; condition (39) would then provide a check on the validity of our theory.

It remains to specify the index of refraction n, or equivalently the quantity $\mu=1-n$. According to [4, p.32], a "canonical" model ocean due to Munk [24] provides a "reasonable description of an oceanic sound channel":

$$\mu(x) = \frac{\epsilon(e^x-x-1)}{1+\epsilon(e^x-x-1)},$$

where ϵ is a small dimensionless parameter ($\epsilon \cong 5.7 \times 10^{-3}$) and the depth x is measured in suitable units (approximately half-kilometers). Since ϵ is small, and x is small for propagation in the sound channel, we neglect the term with ϵ in the denominator, so that

$$\mu(x) = \epsilon(e^x-x-1) \;;$$

using the smallness of x again, we approximate this function by the first term in its Maclaurin series:

$$\mu(x) = \frac{\epsilon}{2}x^2 \;. \tag{40}$$

Using (37) and (38) in (36), we obtain

$$t(z) = \frac{1}{C_o}\int_o^z\Big(1-\mu(x(z))\Big)\left(1+\frac{u^2(x(z),z)+v^2(x(z),z)}{L^2}\right)^{\frac{1}{2}}dz \;.$$

Recalling (35) and (39), using the approximation $(1+w)^{\frac{1}{2}} = 1+\frac{1}{2}w$ for small w, and dropping the product of two small terms, we arrive at

$$t(z) = \frac{1}{C_o}\int_o^z\left(1-\mu(x(z)) + \frac{u^2(x(z),z)+v^2(x(z),z)}{2L^2}\right)dz \;. \tag{41}$$

Equation (41) serves as our definition of the travel time from range 0 to range z along the random ray x(z). Since the travel time $t_o(z)$ for the direct ray is just $t_o(z) = z/C_o$, if we denote by $t_D(z)$ the difference $t_o(z)-t(z)$ between these travel times, we

have from (41), with μ given by (40),

$$t_D(z) = \frac{1}{C_o}\int_o^z\left(\frac{\epsilon}{2}x^2(z) - \frac{1}{2L^2}(u^2(x(z),z) + v^2(x(z),z))\right)dz . \tag{42}$$

IV. <u>Average Travel Time</u>

Let us take the average of equation (42) above; we get

$$E(t_D) = \frac{1}{C_o}\int_o^z(E(\frac{\epsilon}{2}x^2) - \frac{1}{L^2}E(\tfrac{1}{2}u^2+\tfrac{1}{2}v^2))dz ,$$

which, by (35) and (40) becomes

$$E(t_D) = \frac{1}{C_o}\int_o^z(E(\mu) - \frac{1}{L^2}(K-E(\mu))dz ,$$

or

$$E(t_D) = \frac{1}{C_o}\int_o^z\{(-K/L^2) + (1+1/L^2)E(\mu)\}dz . \tag{43}$$

To perform the integration, we need a formula for $E(\mu)$; this is readily obtained from the virial theorem (34), which for μ given by (40) may be written

$$\frac{d^2}{dz^2}E(\mu) = \epsilon E(u^2+v^2) - 2\epsilon E(\mu) .$$

Using (35) again, we find the last equation becomes simply

$$\frac{d^2}{dz^2}E(\mu) + 4\epsilon E(\mu) = 2\epsilon K . \tag{44}$$

Supposing the acoustic source to be at the origin of our coordinate system, and noting that $\frac{d}{dz}E(\mu) = \epsilon E(vx)$, by (28), we have the initial conditions

$$\frac{dE(\mu)}{dz} = E(\mu) = 0 , \quad z=0 . \tag{45}$$

The unique solution to (44)-(45) is

$$E(\mu) = \frac{K}{2}[1 - \cos(2\sqrt{\epsilon}z)] \ . \tag{46}$$

Substituting (46) into (43), performing the integration, and recalling $t_o(z) = z/C_o$, we obtain

$$t_o(z)-E(t(z)) \equiv E(t_D) = \frac{Kt_o(z)}{2}\left\{- \frac{1}{L^2} + [1 - (\frac{\sin(2\sqrt{\epsilon}z)}{2\sqrt{\epsilon}z})]\right\} , \tag{47}$$

which is the formula mentioned in the introduction for the difference between the travel time $t_o(z)$ of the direct ray to range z and the average travel time of all rays to range z.

Let us observe some features of formula (47). First, note that $E(t_D)$ is proportional to t_o, in accord with the well-known rule that pulse spreading in the sound channel is proportional to range ([31], p.102). Also observe that

$$\lim_{z\to o} (\frac{(t_o(z)-E(t(z)))}{t_o(z)}) = \frac{-K}{2L^2} < 0 ,$$

so that the direct ray has shorter than average travel time and therefore arrives early, as in FIG. A above, for relatively small values of the range z. On the other hand, provided

$$L^2 > 1 , \tag{48}$$

we see that $t_o(z)-E(t(z)) > 0$ for large z, so that refracted rays tend to arrive before the direct ray, as in FIG. B. At some intermediate range $t_o(z) = E(t(z))$, and we have a convergence zone. Assuming ϵ is known from properties of seawater (as is the case, at least approximately, according to [4]), equation (47) implies that knowledge of the location of the convergence zone is equivalent to knowledge of the value of L. To check the validity of the theory, we could empirically determine the location of the

convergence zone for pulse propagation along the sound channel axis and use it to determine L from (47) (note that such an L would automatically satisfy (48)). We could then use suitably normalized and averaged records of the type in FIG. B to form an empirical estimate of the probability density function of $t(z)$, and hence of $E(t(z))$. Knowing ϵ and L, we could then estimate K from equation (47) and see whether (39) holds.

Locating the source off the sound channel axis, at depth $x_o \neq 0$, requires initial conditions other than (45): one condition is $E(\mu)=\frac{\epsilon}{2} x_o^2$ at $z=0$, but specifying $E(vx)$ at $z=0$ seems to require the introduction of another undetermined parameter into the initial conditions, and hence into the (readily obtained) analogue of equation (47).

Note that (46) implies that

$$\lim_{z\to\infty} \frac{1}{z} \int_o^z E(\mu(x(z)))dz = \frac{K}{2} , \qquad (49)$$

which, interpreted as a quantum-mechanical statement, is the well-known equipartition of energy for the harmonic oscillator. Comparing equation (46) with the corresponding equation in the usual quantum-mechanical proof of (49) (cf. [32], p.108) yields the fact, not immediately obvious, that the matrix elements of the potential energy between different and non-adjacent eigenstates vanish.

With μ given by (40), Ehrenfest's Theorem (32) and the initial condition $x(0)=0$ yield

$$E(x(z)) = A \sin(\sqrt{\epsilon} z) ,$$

where, by (31), $A = (1/\sqrt{e})E(v(0,0))$. Thus, if there is any difference between the amount of acoustic energy initially directed upward and the amount initially directed downward, so that $E(v(0,0)) \neq 0$, then the acoustic particles on the average execute simple harmonic motion above and below the sound axis.

V. Discussion

We have presented a theory of acoustic pulse propagation in the deep ocean sound channel in terms of the stochastic mechanical interpretation of the parabolic wave equation. We have derived the formula (47) for average travel time along a random ray and shown how it accounts for the transformation of near-range pulse shapes (FIG. A) into far-range pulse shapes (FIG. B). We have also described how the theory can be tested against empirical data.

As mentioned in the introduction, we would like to use the theory to predict the average long-range pulse shape; or, equivalently, to find the asymptotic probability distribution of $t_D(z)$ as $z \to \infty$. It follows from equation (49) and an easy generalization of work of Darling and Kac [33] to non-stationary processes that

$$\frac{1}{z}\int_0^z \mu(x(z))dz \to \frac{K}{2}, \quad z \to \infty,$$

in probability. If we could decompose (u^2+v^2) as

$$u^2 + v^2 = A + B,$$

where, in some suitable far-field approximation,

a) A and B have no explicit dependence on z; i.e., they depend on z only through x(z),

b) $\frac{1}{z}\int_0^z E(A)dz \to C_1$, for some constant C_1,

and c) $\frac{1}{L(z)}\int_0^z E(B)dz \to C_2$, for some constant C_2 and slowly varying function L(z) such that $L(z)\to\infty$ as $z\to\infty$ (for example, L(z)=log z, L(z)=log(log z), etc.),

then equation (42) and the results of Darling and Kac would imply that, if we put $X(z) = \frac{1}{z}\int_0^z B(x(z))dz$, a suitably normalized version of $t_D(z)$ differs from X(z) by a process approaching 0 in probability as $z\to\infty$, and the distribution of a suitably normalized version of X(z) converges to a unit exponential distribution as $z\to\infty$. In view of FIG. B, this would seem an appropriate and desirable result. We do not know whether such a decomposition satisfying a)-c) is possible.

To apply the results of Darling and Kac, or the results of their followers (e.g., [34-38]) in analyzing the asymptotic distribution of functionals of diffusion processes of the form

$$\int_0^z f(x(z))dz ,$$

condition a) is essential. These authors also assume that the process x(z) is temporally homogeneous (i.e., stationary). Because of the explicit dependence of the drift b on z in equation (6), this assumption is not justified in our case. In attempting to apply results like those of [33-38], therefore, one must always inquire whether the authors' arguments hold for non-

stationary processes. The same difficulties are encountered in trying to apply other limit theorems for diffusion processes (e.g., [39], [40]).

It would also be desirable to explore properties of the travel time t(z) of equation (41) with a more realistic choice of μ than (40); the formula preceding (40) seems a natural next step. With this choice of μ we have so far not succeeded in finding an explicit formula for $E(\mu)$ to replace (46). We intend further study of this question, and expect the search for explicit formulas for $E(\mu)$ to involve suitable approximations which we cannot presently anticipate.

We have been avoiding the question of boundary conditions at the sea surface and bottom. Allowing x(z) to vary only in a finite interval, rather than from $-\infty$ to $+\infty$, would involve consideration of the boundary behavior of the diffusion process near the endpoints (e.g., [20], chapter 4) and the attendant analytical complications, and would seem to be of limited physical relevance for long-range pulse propagation in the sound channel. However, certain advantages might well accrue. For example, if the time the particle hits the bottom (or surface) has an exponential distribution (reference [41], chapter V, gives conditions on stationary processes which guarantee this), then taking absorption at the surface and bottom into account by "killing" particles as they reach the boundary introduces the usual exponential attenuation factor for acoustic intensity. In addition, it is conceivable that the extra a priori complications

of a model with finite depth would lead to amelioration of some of the difficulties mentioned earlier--for example, those associated with the prediction of pulse shape.

In the present paper we have used methods which are unfamiliar in acoustics. However, we believe that we have taken at least a small step toward demonstrating that these methods can yield insight into at least some acoustic phenomena amenable to the parabolic approximation. We conclude by pointing out another underwater acoustic phenomenon of current interest to which we believe our methods are well-suited. Recently Dosso and Chapman [42] have studied the leakage of energy from a shallow sound channel in the Pacific Ocean into the deep sound channel. In the parabolic approximation, this phenomenon is identical to particles tunneling through potential barriers in quantum mechanics. Using stochastic mechanical methods, Jona-Lasinio, Martinelli, and Scoppola [43, 44] have recently gained significant new insight into quantum-mechanical tunneling between wells of multi-welled potentials. Following their approach in analyzing the situation studied by Dosso and Chapman promises to lead to another demonstration of the value of stochastic mechanical methods in research in underwater acoustics.

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